

Plastikstufe with toric core

Jiro ADACHI

Abstract

Plastikstufes and overtwistedness for higher-dimensional contact manifolds are studied in this paper. It is proved that a contact structure is overtwisted if and only if there exists a small plastikstufe with toric core that has trivial rotation.

1 Introduction

“Overtwisted” is a remarkable class of contact structures where a parametric h -principle holds. This class for contact structures on higher-dimensional manifolds is introduced recently (see [BEM]) although such a class is introduced for contact structures on 3-dimensional manifolds few decades ago (see [E]). However, compared with the 3-dimensional case, geometric characterization of this notion is still unclear. There are some discussions and proposals in [BEM] and [CMP]. In this paper, a small improvement of one of the ideas in [CMP] is given. The improvement is suitable for the modification of contact structure introduced in [Aj].

Contact structure is a hyperplane field on an odd-dimensional manifold which is completely non-integrable. Borman, Eliashberg, and Murphy proved the existence of a class of contact structures where contact structures homotopic to each other as almost contact structures are isotopic (see [BEM], [E]). A contact structure in the class is said to be *overtwisted*. The class is defined or characterized by the existence of a certain piecewise smooth $2n$ -dimensional disc embedded into a $(2n + 1)$ -dimensional contact manifold. However the definition of such a disc is rather complex although it is comparatively easy in dimension 3 (see Subsections 2.1 and 2.3).

There are some proposals for characterizations of the overtwistedness. A characterization by plastikstufe is given by Casals, Murphy, and Presas [CMP]. A plastikstufe is introduced by Niederkrüger [N] as an obstruction to symplectic fillability. It is known that if a contact structure on a closed 3-manifold is overtwisted then it never appears as a certain boundary of a compact symplectic 4-manifold. In this sense, a plastikstufe is regarded as a higher-dimensional generalization of an overtwisted 2-dimensional disc.

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In [CMP], they impose some conditions on plastikstufes. A *plastikstufe* \mathcal{P} is a certain product $D_{\text{ot}}^2 \times B^{n-1}$ of a simple overtwisted 2-disc D_{ot}^2 with a closed orientable $(n-1)$ -dimensional manifold B^{n-1} embedded into a $(2n+1)$ -dimensional contact manifold (M, ξ) (see Subsection 2.1 for definition). This manifold B^{n-1} is called the *core* of the plastikstufe. By definition, a submanifold of \mathcal{P} corresponding to $(0, 1) \times B^{n-1} \subset D_{\text{ot}}^2 \times B^{n-1}$ is Legendrian, where $(0, 1) \subset D_{\text{ot}}^2$ corresponds to a part of a non-compact leaf of singular foliation on D_{ot}^2 (see Figure 2.1). It is called a *leaf ribbon* of \mathcal{P} . A plastikstufe $\mathcal{P} \subset (M, \xi)$ is said to be *small* if it is contained in an open ball in (M, ξ) . When a plastikstufe \mathcal{P} is small and with spherical core $B^{n-1} = S^{n-1}$, it is said to have the *trivial rotation* if a leaf ribbon of \mathcal{P} is isotopic to a punctured Legendrian disc $\text{int } D^n \setminus \{0\}$.

Then the following is given in [CMP] (The “only if” part is due to the h -principle in [BEM]):

Theorem 1.1 (Casals, Murphy, Presas). *Let (M, ξ) be a contact manifold of dimension $2n+1$. The contact structure ξ is overtwisted if and only if there exists a small plastikstufe with a spherical core S^{n-1} that has trivial rotation.*

Then our interest goes to examples or constructions of such structures on a given manifold. In dimension 3, there exists a famous modification of contact structure, the Lutz twist, that creates overtwisted discs without changing the given manifold. A higher-dimensional generalization of the Lutz twist was introduced by Etnyre and Pancholi [EP], that creates a plastikstufe. It is confirmed in [CMP] that the plastikstufe is small and has trivial rotation.

On the other hand, another higher-dimensional generalization of the Lutz twist is introduced by the author [Aj]. By the modification, we also obtain plastikstufes in the given manifold. However the plastikstufes are with toric cores $B^{n-1} = T^{n-1}$. This is the first motivation of this paper. We introduce, in this paper, rotation class of small plastikstufes with toric core. Then we prove the following:

Theorem A. *Let (M, ξ) be a contact manifold of dimension $2n+1$. The contact structure ξ is overtwisted if and only if there exists a small plastikstufe with a toric core T^{n-1} that has trivial rotation.*

We should remark that it is proved in [Aj] that the modification creates overtwisted discs directly.

This paper is organized as follows. In the next section, we review the important notions: plastikstufe, loose Legendrian submanifold, and overtwistedness. In Section 3, we discuss on rotation class of plastikstufes with toric core. Then Theorem A is proved in Section 4. Last of all, in Section 5, we discuss how to create plastikstufes with toric core and trivial rotation without changing the underlying manifold.

Recently, a stronger result is informed by Huang [H]. It is claimed that any embedded plastikstufe implies overtwistedness. However, it seems it is still important to know concrete shapes of plastikstufes.

2 Preliminaries

In this section, we review some notions and properties needed in the following sections. Plastikstufe (Subsection 2.1), loose Legendrian submanifold (Subsection 2.2), and overtwistedness (Subsection 2.3) are introduced. In Subsection 2.4, we review some results concerning h -principle.

2.1 Plastikstufe

Plastikstufe is an obstruction to symplectic fillability introduced by Niederkrüger [N]. A prototype of this notion was introduced by Gromov [Gr1]. In this subsection, we review the definition and basic properties.

In order to define Plastikstufe, we first introduce an overtwisted disc in a 3-dimensional contact manifold. Let (M, ξ) be a 3-dimensional contact manifold, and $D \subset (M, \xi)$ an embedded disc. The contact structure ξ trace a singular 1-dimensional foliation D_ξ on D called the *characteristic foliation*. The disc D is called an *overtwisted disc* if the characteristic foliation is isomorphic to Figure 2.1(1). The boundary ∂D is a Legendrian circle and the center is a singular point. It is

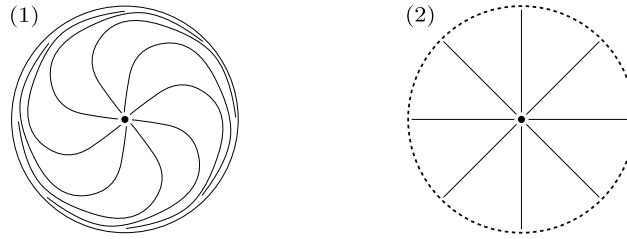


Figure 2.1: overtwisted disc in dimension 3.

obtained from a non-generic disc in Figure 2.1(2) by perturbing slightly. In the second disc, the boundary and the center is the set of singular points.

Then the plastikstufe is defined as follows. Let (M, ξ) be a contact manifold of dimension $2n+1 > 3$, and B a closed manifold of dimension $n-1$.

Definition. A *plastikstufe* with *core* B is a submanifold $\mathcal{P}_B \subset (M, \xi)$ diffeomorphic to $D^2 \times B$ which satisfies the following conditions:

- each fiber $\{z\} \times B$ is tangent to ξ for any $z \in D^2$,
- on each slice $D^2 \times \{b\}$, $\xi \cap T(D^2 \times \{b\})$ generates the same singular foliation as the overtwisted disc (see Figure 2.1(1)) for any $b \in B$.

A plastikstufe is said to be *small* if it is contained in an embedded open ball in (M, ξ) .

The submanifold in a plastikstufe $\mathcal{P}_B \subset (M, \xi)$ corresponding to $(0, 1) \times B \subset D^2 \times B$ is a Legendrian submanifold, where $(0, 1)$ is a leaf of the characteristic foliation on the overtwisted disc (see Figure 2.1). A thin ribbon corresponding to $(0, \varepsilon) \times B \subset (0, 1) \times B$ sufficiently close to the core B is called a *leaf ribbon* of the plastikstufe \mathcal{P}_B . All leaf ribbons are isotopic as Legendrian

submanifolds.

A contact structure ξ is said to be *PS-overtwisted* if there exists a plastikstufe in (M, ξ) .

An important property of plastikstufe is the following due to Niederkrüger [N]:

Theorem 2.1. *If a closed contact manifold has a plastikstufe, then it can not have any (semi-positive) symplectic filling.*

A plastikstufe has a standard neighborhood although it has codimension $n > 1$. Let \mathcal{P}_B be a plastikstufe with core B in a contact manifold (M, ξ) of dimension $2n + 1$. It is proved that there exists the standard tubular neighborhood of $\mathcal{P}_B \subset (M, \xi)$ described as follows (see [MNPS]). Setting $\alpha_{\text{ot}} := \cos rdz + \sin rd\theta$, we have the standard overtwisted contact form α_{ot} on \mathbb{R}^3 with the cylindrical coordinates (z, r, θ) . Let $D_{\text{ot}}^2 \subset (\mathbb{R}^3, \ker \alpha_{\text{ot}})$ denote an overtwisted disc. Setting $\alpha_{\text{PS}} := \alpha_{\text{ot}} + \lambda_{\text{can}}$, where $\lambda_{\text{can}} = \mathbf{q}d\mathbf{p}$ is the canonical Liouville 1-form on T^*B , we have a contact form on $\mathbb{R}^3 \times T^*B$. Let $\xi_{\text{PS}} = \ker \alpha_{\text{PS}}$ denote the corresponding contact structure. Then there exists a tubular neighborhood $U_{\text{PS}} \subset (M, \xi)$ of \mathcal{P}_B and a contact embedding $\varphi_{\text{PS}}: (U_{\text{PS}}, \xi) \rightarrow (\mathbb{R}^3 \times T^*B, \xi_{\text{PS}})$ that maps \mathcal{P}_B to $\varphi_{\text{PS}}(\mathcal{P}_B) = D_{\text{ot}}^2 \times B_0$, where B_0 is the zero-section of T^*B .

2.2 Loose Legendrian submanifold

Loose Legendrian submanifold is a special Legendrian submanifolds in a contact manifold of dimension greater than 3. It is introduced by Murphy [M] as a class where a parametric h -principle holds. In this subsection, we review the definition and basic properties.

First, we define loose Legendrian submanifold. It is defined by using model chart. We introduce some parts of the model. Let ξ_{std} be the standard contact structure on \mathbb{R}^3 , and $\alpha_{\text{std}} := dz - ydx$ the standard contact form defining ξ_{std} . In $(\mathbb{R}^3, \xi_{\text{std}})$, let L_0 be a negatively stabilized Legendrian curve as in Figure 2.2(4). More precisely, it is a part of the curve

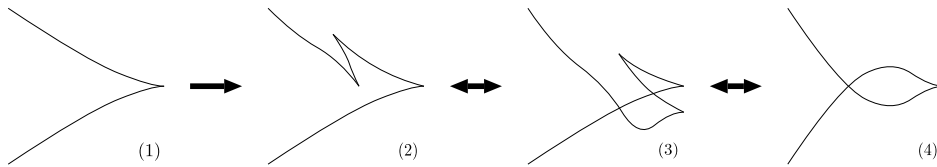


Figure 2.2: Stabilization of a Legendrian arc (front projection).

$$\left(t^2, \frac{15}{4}(t^3 - t), \frac{3}{2}t^5 - \frac{5}{2}t^3 \right)$$

including the cusp and the crossing point in Figure 2.2(4). Let $W \subset (\mathbb{R}^3, \xi_{\text{std}})$ a convex open ball that contains L_0 . Next, we introduce some symplectic parts. Let $\lambda_{\text{can}} = \sum q_i dp_i$ be the standard Liouville form on $T^*\mathbb{R}^{n-1}$ and $Z \subset T^*\mathbb{R}^{n-1}$ the Lagrangian zero section. Set

$$V_\rho := \left\{ (p_1, \dots, p_{n-1}, q_1, \dots, q_{n-1}) \in T^*\mathbb{R}^{n-1} \mid p_1^2 + \dots + p_{n-1}^2 < \rho^2, q_1^2 + \dots + q_{n-1}^2 < \rho^2 \right\}.$$

Then the 1-form $\alpha_{\text{std}} + \lambda_{\text{can}}$ is a contact form on $\mathbb{R}^3 \times T^*\mathbb{R}^{n-1} \cong \mathbb{R}^{2n+1}$, which is nothing but the standard contact structure. We abuse the notation ξ_{std} for $\ker(\alpha_{\text{std}} + \lambda_{\text{can}})$ as well. The submanifold $L_0 \times Z \subset (\mathbb{R}^{2n+1}, \xi_{\text{std}})$ is a Legendrian submanifold.

Now, loose Legendrian submanifold is defined as follows. It is defined by Murphy [M]. The following definition is due to [MNPS], which is equivalent to the original one.

Definition. The relative pair $(W \times V_\rho, L_0 \times Z)$ of an open set and a Legendrian submanifold in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$ for some convex open set $W \subset \mathbb{R}^3$ containing L_0 is called a *loose chart* if $\rho > 1$.

Let (M, ξ) be a contact manifold of dimension $2n + 1 > 3$. A connected Legendrian submanifold $\Lambda \subset (M, \xi)$ is said to be *loose* if there exists an open set $U \subset M$ so that $(U, U \cap \Lambda)$ is contactomorphic to a loose chart.

The notion, loose Legendrian, is introduced as a class that satisfies the parametric h -principle. The following theorem is proved by Murphy [M].

Theorem 2.2. *Let (M, ξ) be a contact manifold of dimension $2n + 1 > 3$. If loose Legendrian submanifolds $\Lambda_0, \Lambda_1 \subset (M, \xi)$ are isotopic as embeddings, then they are isotopic as Legendrian embeddings.*

Some relations between loose Legendrian submanifolds and plastikstufes are studied in [MNPS]. In order to state the result, we need some other notions. We mention the relation in Section 3. A key observation for the results, as well as for results in this paper, is such a relation in dimension 3. A relation between negative stabilization of a Legendrian knot and an overtwisted disc in a contact 3-manifold is discussed also in [MNPS].

Theorem 2.3. *Let (M, ξ) be an overtwisted contact 3-manifold with an overtwisted disc D_{ot}^2 . Suppose that $L \subset (M, \xi)$ is a Legendrian knot which never intersects with the overtwisted disc D_{ot}^2 . Then the Legendrian knot $\tilde{L} := L \# \partial D_{\text{ot}}^2$ obtained as a Legendrian connected sum of L and the boundary ∂D_{ot}^2 of the overtwisted disc is a negative destabilization of L . Further, L and \tilde{L} are isotopic as Legendrian knots in (M, ξ) .*

It is a key idea in the discussion in Subsection 4.2 for the proof of Theorem A.

2.3 Overtwistedness

The notion, overtwistedness of a contact structure, implies a class of contact structures where a parametric h -principle holds. For higher dimensions, it is introduced by Borman, Eliashberg, and Murphy [BEM]. They defined the overtwisted disc in any dimension, and proved that the class of contact structures which have the overtwisted discs satisfies the h -principle (Theorem 2.4). However, as mentioned in Introduction, the definition of the overtwisted disc is complicated. One of the purposes of this paper is to find another characterization than the overtwisted disc. Then we omit the definition of overtwisted disc in this paper. We assume the existence of the class of contact structures that satisfies Theorem 2.4. The key tool to prove the overtwistedness is

Proposition 2.6 below due to Casals, Murphy, and Presas [CMP], that gives a sufficient condition for the overtwistedness in relation to the loose Legendrian submanifolds.

The most important properties of the class, overtwisted contact structures, is that it satisfies the h -principle. It is proved by Borman, Eliashberg, and Murphy [BEM]. In terms of h -principle, the formal counterpart for contact structure is almost contact structure. Let M be a $(2n + 1)$ -dimensional manifold and $A \subset M$ a subset which satisfies that $M \setminus A$ is connected. Let ξ be an almost contact structure on M which is a genuine contact structure on an open neighborhood $Op(A)$ of $A \subset M$. Then let $\mathfrak{Cont}_{ot}(M; A, \xi)$ denote the set of contact structures on M which are overtwisted on $M \setminus A$ and coincide with ξ on $Op(A)$, and $\mathfrak{cont}(M; A, \xi)$ the set of almost contact structures on M which coincide with ξ on $Op(A)$. Further, for an embedding $\phi: D_{ot}^2 \rightarrow M \setminus A$, we introduce the following subsets of the sets above, where D_{ot}^2 is the overtwisted disc with a germ of contact structure. Let $\mathfrak{Cont}_{ot}(M; A, \xi, \phi) \subset \mathfrak{Cont}_{ot}(M; A, \xi)$ and $\mathfrak{cont}_{ot}(M; A, \xi, \phi) \subset \mathfrak{cont}(M; A, \xi)$ denote subsets consist of contact and almost contact structures for each of which ϕ is a contact embedding. Then the following is one of the most important theorems in [BEM].

Theorem 2.4 (Borman, Eliashberg, Murphy [BEM]). *Let $j: \mathfrak{Cont}_{ot}(M; A, \xi) \rightarrow \mathfrak{cont}(M; A, \xi)$ be the inclusion mapping. Then the induced mapping*

$$j_*: \pi_0(\mathfrak{Cont}_{ot}(M; A, \xi)) \rightarrow \pi_0(\mathfrak{cont}(M; A, \xi))$$

is isomorphic. Moreover, the restriction

$$j|_{\mathfrak{Cont}_{ot}(M; A, \xi, \phi)}: \mathfrak{Cont}_{ot}(M; A, \xi, \phi) \rightarrow \mathfrak{cont}_{ot}(M; A, \xi, \phi)$$

is weak homotopy equivalent.

As a corollary, the following is proved for isocontact embedding.

Corollary 2.5 (Borman, Eliashberg, Murphy [BEM]). *Let (M, ξ) be a connected overtwisted contact manifold of dimension $2n + 1$ and (N, ζ) be an open contact manifold of the same dimension. Let $f: N \rightarrow M$ be an embedding covered by a bundle homomorphism $\Phi: TN \rightarrow TM$ which preserves contact structures fiberwise and conformal symplectic structure on contact hyperplanes. If $df: TN \rightarrow TM$ is homotopic to Φ as bundle monomorphisms, then there exists a isocontact embedding $\tilde{f}: (N, \zeta) \rightarrow (M, \xi)$ isotopic to f . In particular, if a contact manifold is overtwisted, any contact open ball of the same dimension can be embedded into it.*

From this corollary, it follows that an overtwisted contact manifold should have a plastikstufe (see [BEM]). This essentially implies the necessary condition of Theorem A. We discuss more precisely in Subsection 4.1.

We should mention a relation between loose Legendrian submanifold and overtwisted contact structure. To do that, we define trivial Legendrian sphere. Let η_0 be the standard contact structure

$$\ker \left(\sum_{i=1}^{n+1} x_i dy_i - y_i dx_i \right) \Big|_{TS^{2n+1}}$$

on the unit sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$, where $(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) \in \mathbb{R}^{2n+2}$ are coordinates. The n -dimensional sphere

$$\Lambda_0 := \{(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) \in S^{2n+1} \mid y_1 = \dots = y_{n+1} = 0\} \subset (S^{2n+1}, \eta_0)$$

is Legendrian. Since $(S^{2n+1}, \eta_0) \setminus \{\text{point}\}$ is contactomorphic to the standard contact space $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$, the Legendrian submanifold Λ_0 is identified with a topologically trivial Legendrian sphere in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$. Then a Legendrian sphere Λ in a contact manifold (M, ξ) of dimension $2n+1$ is said to be *trivial* if there exists a contact embedding of an open ball $U \subset (\mathbb{R}^{2n+1}, \eta_0)$ containing Λ_0 to (M, ξ) which maps Λ_0 to Λ . Using these notions, the following relation between loose Legendrian submanifold and overtwistedness is proved.

Proposition 2.6 (Casals, Murphy, and Presas [CMP]). *Let (M, ξ) be a contact manifold of dimension $2n+1 > 3$. If the trivial Legendrian sphere $\Lambda_0 \subset (M, \xi)$ is loose, then ξ is overtwisted.*

This property is used to show the sufficient condition of Theorem A in Subsection 4.2.

2.4 h -principle

In this section, we review the Smale-Hirsch immersion Theorem, and Gromov's h -principle for subcritical isotropic submanifolds. For the subjects in this subsection, the readers should consult [EM] and [Am] as well as [Gr2].

First, we introduce the Smale-Hirsch immersion theorem. Let M and V be manifolds of dimension n and p , respectively. Let $\text{Imm}(M, V)$ denote a set of all immersions from M to V endowed with C^∞ -topology, and $\text{Mono}(TM, TV)$ a set of all monomorphisms, that is a bundle homomorphisms which are fiberwise injective, from TM to TV endowed with compact-open topology.

Theorem 2.7 (the Smale-Hirsch immersion theorem). *If $n = \dim M < p = \dim V$, then the inclusion*

$$\text{Imm}(M, V) \hookrightarrow \text{Mono}(TM, TV), \quad f \mapsto df$$

is a weak homotopy equivalence.

In other words, the parametric C^0 -dense h -principle holds for such immersions. The existence of a path of monomorphisms implies the existence of a path of immersions.

Next, we introduce Gromov's h -principle for subcritical isotropic embeddings. In this paper, we need a version for contact structures. Let (W, ξ) be a contact manifold of dimension $2n+1$, and V a manifold of dimension $m < n$. Let $\text{Emb}_{\text{isot}}(V, W)$ denote a set of all isotropic embeddings from V to (W, ξ) , and $\text{Mono}_{\text{isot}}^{\text{emb}}(TV, TW)$ a set of all isotropic monomorphisms, that is a bundle homomorphisms which are fiberwise isotropic injective, from TV to (TW, ξ) .

Theorem 2.8 (Gromov). *The inclusion*

$$\text{Emb}_{\text{isot}}(V, W) \hookrightarrow \text{Mono}_{\text{isot}}^{\text{emb}}(TV, TW), \quad f \mapsto df$$

is a weak homotopy equivalence.

In other words, the parametric C^0 -dense h -principle holds for such embeddings. The existence of a formal isotopy of subcritical isotropic embeddings implies the existence of a path of isotropic embeddings.

3 Rotation class of plastikstufe

Rotation class of a plastikstufe is introduced in Subsection 3.1. In order to discuss plastikstufe with toric core, trivial rotation for such plastikstufe is introduced in Subsection 3.2.

3.1 Rotation class and loose Legendrian submanifold

We introduce the rotation class of a plastikstufe. Then we introduce some results on the relation between plastikstufes with spherical core and loose Legendrian submanifolds.

The rotation class of a plastikstufe is defined as the relative rotation class of the leaf ribbon of the plastikstufe with respect to the “standard” Legendrian ribbon. It is defined for the small plastikstufes with spherical core by Murphy, Niederkrüger, Plamenevskaya, and Stipsicz in [MNPS]. Then we first introduce the relative rotation class of two Legendrian immersions. Then we define rotation class of plastikstufes with spherical core following [MNPS]. In Subsection 3.2, we introduce such notion for plastikstufes with “toric” core.

We define the relative rotation class of Legendrian immersions. Let (M, ξ) be a contact manifold of dimension $2n + 1$ and α a contact form defining ξ . And let $f, g: \Lambda \rightarrow (M, \xi)$ be two Legendrian immersions. Assume that there exists an open ball $U \subset M$ that contains both of the image $f(\Lambda)$, $g(\Lambda)$. First, we regard the contact structure $\xi|_U$ restricted to U as the trivial complex vector bundle as follows. Let J be an almost complex structure on ξ compatible with the conformal symplectic structure induced by $d\alpha|_\xi$. By taking a J -complex trivialization of (ξ, J) , we can regard $(\xi|_U, J)$ as the trivial bundle $\mathbb{C}^n \times U \rightarrow U$. Then, by this identification, df can be regarded as the bundle map $df: T\Lambda \rightarrow \Lambda \times \mathbb{C}^n$, for which $(df)_x(T_x\Lambda) \subset (\xi_{f(x)}, J)$ is totally real since $f: \Lambda \rightarrow (M, \xi)$ is Legendrian. Therefore, we have the complexification

$$df^{\mathbb{C}}: T\Lambda \otimes \mathbb{C} \rightarrow \Lambda \times \mathbb{C}^n$$

which is fiberwise complex isomorphism. For another Legendrian immersion $g: \Lambda \rightarrow (M, \xi)$, we also have a complex bundle map $dg^{\mathbb{C}}: T\Lambda \otimes \mathbb{C} \rightarrow \Lambda \times \mathbb{C}^n$ which is fiberwise complex isomorphism. Then, from the two maps $df^{\mathbb{C}}$ and $dg^{\mathbb{C}}$, a mapping $\varphi: \Lambda \rightarrow \text{GL}(n, \mathbb{C})$ is determined by

$$(dg^{\mathbb{C}})_x = \varphi(x) \circ (df^{\mathbb{C}})_x, \quad x \in \Lambda.$$

Now, we define the *relative rotation class* of g with respect to f as the homotopy class of φ in $[\Lambda, \text{GL}(n, \mathbb{C})] \cong [\Lambda, U(n)]$.

The rotation class of a plastikstufe is defined as follows. Let $\mathcal{P}_{S^{n-1}}$ be a small plastikstufe with spherical core S^{n-1} in a contact manifold (M, ξ) of dimension $2n + 1$. Then a leaf ribbon of $\mathcal{P}_{S^{n-1}}$ is a Legendrian submanifold of (M, ξ) diffeomorphic to $(0, 1) \times S^{n-1}$. Let $f: (0, 1) \times S^{n-1} \rightarrow (M, \xi)$

be the Legendrian embedding corresponding to the leaf ribbon. Since $\mathcal{P}_{S^{n-1}}$ is small, we can apply the discussion above. On the other hand, let $D \subset (M, \xi)$ be a Legendrian disc. Note that all Legendrian discs are isotopic as Legendrian submanifolds. Then we take $\text{int } D \setminus \{\text{point}\} \subset (M, \xi)$ as the standard Legendrian submanifold diffeomorphic to $(0, 1) \times S^{n-1}$. Let $g: (0, 1) \times S^{n-1} \rightarrow (M, \xi)$ be the Legendrian embedding corresponding to the standard Legendrian $(0, 1) \times S^{n-1}$. Now, the *rotation class* of the small plastikstufe $\mathcal{P}_{S^{n-1}}$ is defined as the relative rotation class of f with respect to g . The small plastikstufe $\mathcal{P}_{S^{n-1}}$ is said to have *trivial rotation* if the rotation class vanishes.

Using this notion, rotation class, a relation between plastikstufe and loose Legendrian submanifold is proved.

Proposition 3.1 (Murphy, Niederkrüger, Plamenevskaya, Stipsicz [MNPS]). *Let $\mathcal{P}_{S^{n-1}}$ be a small plastikstufe with spherical core and trivial rotation in a contact manifold (M, ξ) of dimension $2n+1 > 3$. Then any Legendrian submanifold $\Lambda \subset (M, \xi)$ disjoint from $\mathcal{P}_{S^{n-1}}$ is loose.*

Combining this with Proposition 2.6, the following relation between plastikstufes and overtwistedness follows as a corollary.

Corollary 3.2 (Casals, Murphy, Presas [CMP]). *Let (M, ξ) be a contact manifold of dimension $2n+1 > 3$. If there exists a small plastikstufe $\mathcal{P}_{S^{n-1}} \subset (M, \xi)$ with spherical core and trivial rotation then the contact structure ξ is overtwisted.*

3.2 Trivial rotation of a plastikstufe with toric core

The rotation class of a plastikstufe with toric core is defined in this subsection. Recall that the rotation class of a plastikstufe with spherical core is defined, in the previous subsection, as a relative rotation class with respect to the punctured Legendrian disc. Instead of that Legendrian submanifold, we need the “standard” Legendrian submanifold for plastikstufes with toric core. We first define such Legendrian submanifold, then define the rotation class.

First of all, we observe what the “standard” Legendrian submanifold should be. Let (M, ξ) be a contact manifold of dimension $2n+1$. Recall that the “standard” Legendrian submanifold for plastikstufes with spherical core S^{n-1} is the punctured Legendrian disc $\text{int } D^n \setminus \{\text{point}\}$ (see Subsection 3.1). In other words, it is a Legendrian submanifold diffeomorphic to the leaf ribbon $(0, 1) \times S^{n-1}$ which is unique up to isotopy (see Subsection 2.1 for definition of leaf ribbon). Therefore, for plastikstufe with toric core T^{n-1} , we need a Legendrian submanifold diffeomorphic to the leaf ribbon $(0, 1) \times T^{n-1}$ which is unique up to isotopy.

We define the “standard” Legendrian submanifold diffeomorphic to $(0, 1) \times T^{n-1}$ in a contact manifold (M, ξ) of dimension $2n+1 > 3$. If $n = 2$ then $T^{n-1} = T^1 = S^1$. Then the definition is the same as the spherical case. We assume $n > 2$. Let $D^n \subset (M, \xi)$ be a Legendrian disc and $T^{n-2} \subset D^n$ a trivially embedded $(n-2)$ -dimensional torus. In other words, $T^{n-2} \subset D^n$ is unknotted (i.e. the boundary of a handlebody). Note that the choice of $T^{n-2} \subset (M, \xi)$ is unique

up to isotopy as isotropic submanifolds. Let $U \subset D^n$ be a tubular neighborhood of T^{n-2} , which is diffeomorphic to $T^{n-2} \times D^2$. According to this identification, set

$$\tilde{U} := T^{n-2} \times (\text{int } D^2 \setminus \{0\}) \subset D^n \subset (M, \xi).$$

Then $\tilde{U} \subset (M, \xi)$ is a Legendrian submanifold diffeomorphic to $(0, 1) \times T^{n-1}$ unique up to isotopy.

Now, we define rotation class for plastikstufe with toric core. Let $\mathcal{P}_{T^{n-1}}$ be a small plastikstufe with toric core in a contact manifold (M, ξ) of dimension $2n + 1 > 3$.

Definition 3.3. The *rotation class* of the small plastikstufe $\mathcal{P}_{T^{n-1}}$ is the relative rotation class of the Legendrian embedding $f: (0, 1) \times T^{n-1} \rightarrow (M, \xi)$ for a leaf ribbon of $\mathcal{P}_{T^{n-1}}$ with respect to the Legendrian embedding $g: (0, 1) \times T^{n-1} \rightarrow (M, \xi)$ for the standard Legendrian submanifold \tilde{U} . The small plastikstufe $\mathcal{P}_{T^{n-1}}$ is said to have *trivial rotation* if the rotation class vanishes.

4 Proof of Theorem A

Theorem A is proved in this section. The ideas of the proof are debt to [MNPS], [BEM], and [CMP]. We show the necessity (“if” part) in Subsection 4.1, and the sufficiency (“only if” part) in Subsection 4.2.

4.1 Overtwistedness implies the existence of a plastikstufe

In this subsection, assuming the overtwistedness, we show the existence of a small plastikstufe with toric core whose rotation is trivial. In other words, we show the following.

Proposition 4.1. *Let (M, ξ) be a contact manifold of dimension $2n + 1 > 3$. If the contact structure ξ is overtwisted, then there exists a small plastikstufe with toric core that has trivial rotation.*

The existence of a plastikstufe is discussed in [BEM] as an existence of a contact embedding of the model plastikstufe, which is defined as follows. In other words, it is the same as the standard tubular neighborhood of a plastikstufe (see Subsection 2.1). Let B be a closed manifold of dimension $n - 1$. Then we have a contact manifold $(\mathbb{R}^3 \times T^*B, \ker(\alpha_{\text{ot}} + \lambda_{T^*B}))$ of dimension $2n + 1$, where $\alpha_{\text{ot}} = \cos rdz + \sin rd\theta$ is the standard overtwisted contact form on \mathbb{R}^3 with the cylindrical coordinates (r, θ, z) , and $\lambda_{T^*B} = \sum p_i dq_i$ is the canonical Liouville form on T^*B with coordinates (q_i, p_i) . In this contact manifold, the submanifold $D_{\text{ot}}^2 \times B_0$ is a plastikstufe with core B , where $D_{\text{ot}}^2 \subset (\mathbb{R}^3, \ker \alpha_{\text{ot}})$ is a simple overtwisted disc, and $B_0 \cong B \subset T^*B$ is the zero-section. Then let (\mathcal{P}_B, ζ) denote the pair of germs of $(2n + 1)$ -dimensional manifold and contact structure along the plastikstufe $D_{\text{ot}}^2 \times B_0 \subset \mathbb{R}^3 \times T^*B$. It is called the *model plastikstufe with core B* . The following is obtained as a corollary of Theorem 2.4 via Corollary 2.5.

Corollary 4.2 (Borman, Eliashberg, Murphy [BEM]). *Let (M, ξ) be an overtwisted contact manifold of dimension $2n + 1 > 3$, and (\mathcal{P}_B, ζ) the model plastikstufe with core B . If the complexification $TB \otimes \mathbb{C}$ of the tangent bundle of B is trivial, then there exists a contact embedding of (\mathcal{P}_B, ζ) into (M, ξ) .*

Now, assuming the overtwistedness, we show the existence of a small plastikstufe with toric core that has trivial rotation.

Proof of Proposition 4.1. Let (M, ξ) be an overtwisted contact manifold of dimension $2n+1 > 3$, and $(\mathcal{P}_{T^{n-1}}, \zeta)$ the model plastikstufe with toric core T^{n-1} . As the tangent bundle $T(T^{n-1})$ is trivial, its complexification $T(T^{n-1}) \otimes \mathbb{C}$ is also trivial. Then, by Corollary 4.2, the model plastikstufe $(\mathcal{P}_{T^{n-1}}, \zeta)$ can be embedded into (M, ξ) . In the proof of Corollary 4.2 in [BEM], the embedding is constructed by h -principle using the Darboux chart. Therefore, $(\mathcal{P}_{T^{n-1}}, \zeta)$ is embedded into an open ball in (M, ξ) . Then the plastikstufe is small.

It remains to show that the embedded plastikstufe has trivial rotation. In order to discuss the rotation class of the embedded plastikstufe, we should observe the embedded image of a leaf ribbon of the model plastikstufe. Then, we should review the construction of the contact embedding of the model plastikstufe in the proof of Corollary 4.2 in [BEM]. The contact embedding of the model plastikstufe $(\mathcal{P}_{T^{n-1}}, \zeta)$ into (M, ξ) is constructed by h -principle from two contact bundle-mappings:

$$\begin{aligned}\Psi: (T(\mathbb{R}^3 \times T^*T^{n-1}), \ker(\alpha_{\text{ot}} + \lambda_{T^*T^{n-1}})) &\rightarrow (T(\mathbb{R}^3 \times T^*T^{n-1}), \ker(\alpha_{\text{st}} + \lambda_{T^*T^{n-1}})), \\ \Phi: (T(\mathbb{R}^3 \times T^*T^{n-1}), \ker(\alpha_{\text{st}} + \lambda_{T^*T^{n-1}})) &\rightarrow (T\mathbb{R}^{2n+1}, \xi_{\text{st}}),\end{aligned}$$

where $\alpha_{\text{st}} = d\phi + \sum r^2 d\theta$ is the standard contact form on \mathbb{R}^3 , and $\xi_{\text{st}} = \ker\{d\phi + \sum r_i^2 d\theta_i\}$ is the standard contact structure on \mathbb{R}^{2n+1} . Note that the first bundle-mapping is on the identity of $\mathbb{R}^3 \times T^*T^{n-1}$, and that the second one is on the mapping $\mathbb{R}^3 \times T^*T^{n-1} = \mathbb{R}^3 \times (T^*S^1)^{n-1} \rightarrow \mathbb{R}^3 \times \mathbb{C}^{n-1} = \mathbb{R}^{2n+1}$. In general the second one is constructed, under the condition that $TQ \otimes \mathbb{C}$ is trivial, by using Gromov's h -principle. In this case, it is constructed from the explicit mapping from T^*S^1 to $\mathbb{C} \setminus \{0\}$. From these Φ, Ψ , and the Darboux chart, we have a contact bundle homomorphism $(T(\mathbb{R}^3 \times T^*T^{n-1}), \ker(\alpha_{\text{ot}} + \lambda_{T^*T^{n-1}})) \rightarrow (TM, \xi)$. Then, by Corollary 2.5, we have a contact embedding of $(\mathbb{R}^3 \times T^*T^{n-1}, \ker\{\alpha_{\text{ot}} + \lambda_{T^*T^{n-1}}\})$ into (M, ξ) that is isotopic to the mapping between the bases of the bundle homomorphisms above. The contact embedding of the model plastikstufe is obtained as a restriction of the mapping to the model plastikstufe $\mathcal{P}_{T^{n-1}} = D_{\text{ot}}^2 \times T_0^{n-2} \subset \mathbb{R}^3 \times T^*T^{n-1}$.

By observing the contact embedding above, it is proved as follows that the embedded model plastikstufe has trivial rotation. We show that an embedded leaf ribbon is the standard Legendrian submanifold diffeomorphic to $(0, 1) \times T^{n-1}$ in the sense of Subsection 3.2. The model plastikstufe $\mathcal{P}_{T^{n-1}} = D_{\text{ot}}^2 \times T_0^{n-1}$ in $(\mathbb{R}^3 \times T^*T^{n-1}, \ker\{\alpha_{\text{ot}} + \lambda_{T^*T^{n-1}}\})$ can be regarded as

$$\begin{aligned}\mathcal{P}_{T^{n-1}} &= D_{\text{ot}}^2 \times S_0^1 \times T_0^{n-2} \subset (\mathbb{R}^3 \times T^*S^1 \times T^*T^{n-2}, \ker\{\alpha_{\text{ot}} + \lambda_{T^*S^1} + \lambda_{T^*T^{n-2}}\}), \\ &= D_{\text{ot}}^2 \times S_0^1 \times (S_0^1 \times \cdots \times S_0^1) \subset (\mathbb{R}^3 \times T^*S^1 \times \cdots \times T^*S^1, \ker\{\alpha_{\text{ot}} + \lambda_{T^*S^1} + \cdots + \lambda_{T^*S^1}\}).\end{aligned}$$

where $S_0^1 \subset T^*S^1$ and $T_0^{n-2} \subset T^*T^{n-2}$ are zero-sections. Then a leaf ribbon is $(0, 1) \times S_0^1 \times T_0^{n-2} \subset \mathcal{P}_{T^{n-1}}$, where $(0, \varepsilon) \subset D_{\text{ot}}^2$ is an open segment on a leaf of the characteristic foliation on D_{ot}^2 (see Figure 2.1). We should recall that each factor T^*S^1 is mapped to $\mathbb{C} \setminus \{0\} \subset \mathbb{C}$ by the mapping

Φ above. Then the image of T^{n-2} is unknotted. Similarly, the image of $(0, 1) \times S_0^1$ is isotopic to $\text{int } D^2 \setminus \{0\} \subset \mathbb{C} \setminus \{0\} \subset \mathbb{C}$. Then we conclude, after applying the h -principle, that the embedded leaf ribbon is the standard Legendrian submanifold in (M, ξ) . In other words, the embedded model plastikstufe has trivial rotation. \square

4.2 The existence of a plastikstufe implies overtwistedness

In this subsection, assuming the existence of a small plastikstufe with toric core and trivial rotation, we show the overtwistedness. The proof is largely debt to the relation between Loose Legendrian submanifold and overtwistedness, Proposition 2.6 due to [CMP]. The main contribution of this paper is the relation between plastikstufe with toric core and loose Legendrian submanifolds, Theorem 4.3 bellow. Combining these results, we obtain the main issue Proposition 4.6 of this subsection in Subsubsection 4.2.2.

4.2.1 Plastikstufe to loose Legendrian

We prove the following claim on a relation between plastikstufes with toric core and loose Legendrian submanifolds.

Theorem 4.3. *Let (M, ξ) be a contact manifold of dimension $2n + 1 > 3$. Suppose that there exists a small plastikstufe $\mathcal{P} \subset (M, \xi)$ with toric core and trivial rotation. Then any Legendrian submanifold which is disjoint from \mathcal{P} is loose.*

In order to prove this theorem, we need the following lemma. It is a key to the proof of Theorem 4.3. For the statement, we recall that a plastikstufe \mathcal{P} in a contact manifold (M, ξ) has the standard tubular neighborhood $\varphi_{\text{PS}}: (U_{\text{PS}}, \xi) \rightarrow (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{PS}})$ for which $\varphi_{\text{PS}}(\mathcal{P}) = D_{\text{ot}}^2 \times T_0^{n-1}$, where $T_0^{n-1} \subset T^*T^{n-1}$ is the zero section (see Subsection 2.1).

Lemma 4.4. *Let (M, ξ) be a contact manifold of dimension $2n+1$, and $\mathcal{P} \subset (M, \xi)$ a small plastikstufe with toric core that has trivial rotation. Then, for any Legendrian submanifold $\Lambda \subset (M, \xi)$ disjoint from the plastikstufe \mathcal{P} , there exist a Legendrian submanifold $\Lambda_0 \subset \Lambda$ diffeomorphic to $(0, 1) \times T^{n-1}$ and an ambient contact isotopy*

$$\varphi: (M, \xi) \times [0, 1] \rightarrow (M, \xi)$$

that satisfy the following conditions: setting $\varphi_t(\cdot) := \varphi(\cdot, t)$,

- $\varphi_t = \text{id}$ near \mathcal{P} for any $t \in [0, 1]$,
- $\varphi_0 = \text{id}$,
- $\varphi_1(\Lambda_0)$ lies in a standard tubular neighborhood $U_{\text{PS}} \subset (M, \xi)$ of the plastikstufe \mathcal{P} , and is diffeomorphic to $(0, 1) \times T_0^{n-1}$ by $\varphi_{\text{PS}}: (U_{\text{PS}}, \xi) \rightarrow (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{PS}})$, where $(0, 1) \subset (\mathbb{R}^3, \xi_{\text{ot}})$ is a Legendrian segment on an extended overtwisted disc, and $T_0^{n-1} \subset T^*T^{n-1}$ is the zero section.

Proof. First of all, we arrange the situation for this proof. A candidate of Λ_0 is given as follows. Since the plastikstufe $\mathcal{P} \subset (M, \xi)$ is small, there exists an open ball $U \subset (M, \xi)$ that includes \mathcal{P} . We may assume that U intersects with the given Legendrian submanifold $\Lambda \subset (M, \xi)$. Taking a Legendrian disc $D_L^n \subset \Lambda \cap U \subset (M, \xi)$, we have, in D_L^n , the standard Legendrian submanifold diffeomorphic to $(0, 1) \times T^{n-1}$ by the construction in Subsection 3.2. Let Λ_0 denote the Legendrian submanifold, and $f_0: (0, 1) \times T^{n-1} \rightarrow (M, \xi)$ be the corresponding Legendrian embedding: $\text{Im } f_0 = \Lambda_0$. On the other hand, fix a Legendrian strip $\Lambda_1 \subset (M, \xi)$, diffeomorphic to $(0, 1) \times T^{n-1}$, which is isotopic to a leaf ribbon of \mathcal{P} but disjoint from \mathcal{P} . Let $f_1: (0, 1) \times T^{n-1} \rightarrow (M, \xi)$ be a corresponding Legendrian embedding. We are going to connect these two Legendrian embeddings f_0, f_1 by a path of Legendrian embeddings. And then it is extended to an ambient contact isotopy.

The proof is divided into the following three steps. We first concentrate on the core-tori of Legendrian submanifolds diffeomorphic to $(0, 1) \times T^{n-1}$, that are subcritical isotropic submanifolds. In Step 1, we connect the restrictions of f_0 and f_1 to the core-tori by a path of subcritical isotropic embeddings. Then, in Step 2, we extend it to a path of Legendrian embeddings. And then it is extended to an ambient contact isotopy in Step 3.

Step 1: We will find the path of subcritical isotropic embeddings using Gromov's h -principle for subcritical isotropic embeddings (Theorem 2.8). In order to apply Gromov's h -principle, we should construct a formal isotopy. Further, the needed formal isotopy is obtained by using The Smale-Hirsch immersion theorem (Theorem 2.7).

What we should do first is to construct a formal monomorphism for Theorem 2.7. Setting $f_i^{\text{cr}} := f_i|_{T^{n-1} \times \{c\}}$, $i = 0, 1$, for some $c \in (0, 1)$, we have two subcritical isotropic embeddings of a torus T^{n-1} :

$$f_i^{\text{cr}}: T^{n-1} \rightarrow (M, \xi).$$

We extend these embeddings to a certain embedding $T^{n-1} \times [0, 1] \rightarrow (M, \xi)$ by Theorem 2.7. To apply it, what we need is a formal monomorphism $T(T^{n-1} \times [0, 1]) \rightarrow TM$. We construct a Lagrangian monomorphism as a real part of a complexification as follows.

First, we extend from the both ends a little. As restrictions of f_0 and f_1 , we have two Legendrian embeddings

$$f_0: T^{n-1} \times [0, \delta] \rightarrow (M, \xi), \quad f_1: T^{n-1} \times [1 - \delta, 1] \rightarrow (M, \xi),$$

reparameterizing $(0, 1)$ so that $f_i|_{T^{n-1} \times \{i\}} = f_i^{\text{cr}}$, $i = 0, 1$, for some small $\delta > 0$. Taking complexifications of tangent bundles and complex trivialization of ξ on U (see Subsection 3.1), we have the following bundle mappings

$$df_0^{\mathbb{C}}: T(T^{n-1} \times [0, \delta]) \otimes \mathbb{C} \rightarrow \mathbb{C}^n, \quad df_1^{\mathbb{C}}: T(T^{n-1} \times [1 - \delta, 1]) \otimes \mathbb{C} \rightarrow \mathbb{C}^n, \quad (4.1)$$

which are fiberwise complex isomorphic. We remark that targets \mathbb{C}^n of the mappings above are fibers of $U \times \mathbb{C}^n \rightarrow U$, the trivialization of $\xi|_U$. The mapping $df_0^{\mathbb{C}}$ is homotopic to the bundle mapping $G_0^{\mathbb{C}}: T(T^{n-1} \times [0, \delta]) \otimes \mathbb{C} \rightarrow \mathbb{C}^n$ defined by

$$(G_0^{\mathbb{C}})_{(x,t)} := (df_0^{\mathbb{C}})_{(x,\delta)}: (T_x(T^{n-1}) \times \mathbb{R}) \otimes \mathbb{C} \rightarrow \mathbb{C}^n, \quad 0 \leq t \leq \delta.$$

Similarly, The mapping $df_1^{\mathbb{C}}$ is homotopic to the bundle mapping $G_1^{\mathbb{C}}: T(T^{n-1} \times [1-\delta, 1]) \otimes \mathbb{C} \rightarrow \mathbb{C}^n$ defined by

$$(G_1^{\mathbb{C}})_{(x,t)} := (df_1^{\mathbb{C}})_{(x,1-\delta)}: (T_x(T^{n-1}) \times \mathbb{R}) \otimes \mathbb{C} \rightarrow \mathbb{C}^n, \quad 1-\delta \leq t \leq 1.$$

Next, we extend the mappings $G_i^{\mathbb{C}}$, $i = 0, 1$, above to the whole $T(T^{n-1} \times [0, 1]) \otimes \mathbb{C}$. These $G_i^{\mathbb{C}}$, $i = 0, 1$, are independent of the choices of $t \in [0, \delta]$ or $t \in [1-\delta, 1]$ respectively, and are also fiberwise complex isomorphic. Then, for a mapping $\psi: T^{n-1} \rightarrow \text{GL}(n, \mathbb{C})$ satisfying $(df_0^{\mathbb{C}})_{(x,\delta)} = \psi(x) \cdot (df_1^{\mathbb{C}})_{(x,1-\delta)}$, we have

$$(G_0^{\mathbb{C}})_{(x,t)} = \psi(x) \cdot (G_1^{\mathbb{C}})_{(x,t)}.$$

On the other hand, the plastikstufe \mathcal{P} has trivial rotation, and a Legendrian strip $\Lambda'_1 = f_1(T^{n-1} \times (0, \delta))$ is isotopic to a leaf-ribbon of \mathcal{P} . In addition, $\Lambda'_0 = f_0(T^{n-1} \times (1-\delta, 1))$ is the standard Legendrian submanifold. Then the mapping ψ is homotopic to the constant mapping $e: T^{n-1} \rightarrow \{e\} \subset \text{GL}(n, \mathbb{C})$ to the identity $e \in \text{GL}(n, \mathbb{C})$. By this homotopy, we have a homotopy $df_t^{\mathbb{C}}$ between $df_0^{\mathbb{C}}$ and $df_1^{\mathbb{C}}$ through fiberwise complex isomorphisms (see [MNPS]). Then we can construct the bundle mapping $G^{\mathbb{C}}: T(T^{n-1} \times [0, 1]) \otimes \mathbb{C} \rightarrow \mathbb{C}^n$ as

$$G_{(x,t)}^{\mathbb{C}} := \begin{cases} (df_0^{\mathbb{C}})_{(x,c(t))} = (df_0^{\mathbb{C}})_{(x,\delta)} = (G_0^{\mathbb{C}})_{(x,t)} & (0 \leq t \leq \varepsilon) \\ (df_t^{\mathbb{C}})_{(x,(1-c(t))\delta+c(t)(1-\delta))} & (\varepsilon \leq t \leq 1-\varepsilon) \\ (df_1^{\mathbb{C}})_{(x,c(t))} = (df_1^{\mathbb{C}})_{(x,1-\delta)} = (G_1^{\mathbb{C}})_{(x,t)} & (1-\varepsilon \leq t \leq 1), \end{cases}$$

where $c: [0, 1] \rightarrow [0, 1]$ is some smooth function satisfying the following conditions for some small $\varepsilon > 0$: (i) $c(t) = 0$, for $t \in [0, \varepsilon]$, (ii) $c(t) = 1$, for $t \in [1-\varepsilon, 1]$. We should mention that the mapping $G^{\mathbb{C}}$ is fiberwise complex isomorphic. Remark that $G^{\mathbb{C}}$ coincides with $G_0^{\mathbb{C}}$ on a neighborhood of $T^{n-1} \times \{0\} \subset T^{n-1} \times [0, 1]$, and with $G_1^{\mathbb{C}}$ on a neighborhood of $T^{n-1} \times \{1\}$.

By taking the real part of $G^{\mathbb{C}}$, we have a bundle mapping

$$\begin{array}{ccc} G: & T(T^{n-1} \times [0, 1]) & \longrightarrow \xi \\ & \downarrow & \downarrow \\ & T^{n-1} \times [0, 1] & \longrightarrow M. \end{array} \quad (4.2)$$

It is a Lagrangian monomorphism that coincides with df_0 and df_1 on a neighborhood of $T^{n-1} \times \{0\}$, $T^{n-1} \times \{1\} \subset T^{n-1} \times [0, 1]$, respectively.

Then we apply the Smale-Hirsch immersion theorem (Theorem 2.7). On account of Theorem 2.7, the existence of the bundle monomorphism G implies the existence of an immersion $g: T^{n-1} \times [0, 1] \rightarrow M$ for which $dg: T(T^{n-1} \times [0, 1]) \rightarrow TM$ is homotopic to G . In addition, we may assume that $g = f_0, f_1$ on neighborhoods of $T^{n-1} \times \{0\}$, $T^{n-1} \times \{1\} \subset T^{n-1} \times [0, 1]$, respectively. Furthermore, by a slight perturbation of g as immersions, we obtain an embedding $\tilde{g}: T^{n-1} \times [0, 1] \rightarrow M$. In fact, the dimension of $T^{n-1} \times [0, 1]$ is n , that of M is $2n+1$, and $n > 1$. By taking the perturbation fixing near the end $T^{n-1} \times \{0, 1\}$, we still may assume that $\tilde{g} = f_0, f_1$

on neighborhoods of $T^{n-1} \times \{0\}$, $T^{n-1} \times \{1\} \subset T^{n-1} \times [0, 1]$, respectively. Then, setting $\tilde{g}_t := \tilde{g}(\cdot, t)$, we obtain a family $\tilde{g}_t: T^{n-1} \rightarrow M$ of embeddings which satisfies $\tilde{g}_0 = f_0^{\text{cr}}$ and $\tilde{g}_1 = f_1^{\text{cr}}$.

The obtained isotopy $\tilde{g}_t: T^{n-1} \times [0, 1] \rightarrow M$ is what we need to apply Gromov's h -principle. In fact, setting $G_t(\cdot) := G(\cdot, t)|_{T^{n-1} \times \{t\}}$, we have a family bundle mapping

$$\begin{array}{ccc} G_t: & T(T^{n-1}) & \longrightarrow \xi \\ & \downarrow & \downarrow \\ & T^{n-1} & \xrightarrow{\tilde{g}_t} M. \end{array}$$

It is a family of isotropic monomorphisms covering \tilde{g}_t which satisfies $G_t = df_0|_{T(T^{n-1} \times \{t\})}$ for $t \in [0, 1]$ close to 0, and $G_t = df_1|_{T(T^{n-1} \times \{t\})}$ for $t \in [0, 1]$ close to 1, and that G_t is homotopic to $d\tilde{g}_t$. In other words, it is a formal isotopy between subcritical isotropic embeddings $f_0^{\text{cr}}, f_1^{\text{cr}}: T^{n-1} \rightarrow (M, \xi)$.

Now, we apply Gromov's h -principle (Theorem 2.8) to obtain a path connecting subcritical isotropic embeddings $f_0^{\text{cr}}, f_1^{\text{cr}}$. By Theorem 2.8, we have a family

$$\tilde{f}_t: T^{n-1} \rightarrow (M, \xi)$$

of subcritical isotropic embeddings that satisfies $\tilde{f}_0 = f_0^{\text{cr}}, \tilde{f}_1 = f_1^{\text{cr}}$.

Step 2: Next, we extend \tilde{f}_t to a family of Legendrian embeddings of $T^{n-1} \times [0, 1]$. First, we endow \tilde{f}_t with symplectically normal “framings”. Then, using the framings, we extend \tilde{f}_t to Legendrian embeddings of $T^{n-1} \times [0, 1]$.

As the framings, we construct a family X_t of nowhere vanishing vector fields defined along isotropic submanifolds $\tilde{f}_t(T^{n-1}) \subset (M, \xi)$ which are tangent to the conformal symplectic normal bundle $\text{CSN}(\tilde{f}_t(T^{n-1}), M)$. The conformal symplectic normal bundle is defined as

$$\text{CSN}(\tilde{f}_t(T^{n-1}), M) := T\{\tilde{f}_t(T^{n-1})\}^{\perp'} / T\{\tilde{f}_t(T^{n-1})\} \subset \xi,$$

where the symbol \perp' stands for the skew orthogonal subspace with respect to the symplectic form $d\alpha$ on $\xi = \ker \alpha$. In other words, X_t are chosen so that $X_t \oplus T\{\tilde{f}_t(T^{n-1})\}$ are Legendrian.

Around the ends $\{0, 1\} \subset [0, 1]$, we already have such framings. In fact, the original embeddings $f_0, f_1: T^{n-1} \times [0, 1] \rightarrow (M, \xi)$ are Legendrian. Then the vector field $df_i(\partial/\partial s)$, $i = 0, 1$, for coordinate $s \in [0, 1]$ are the required framings. Let X_i , $i = 0, 1$, denote such framings.

We construct a family X_t of framings connecting X_0 and X_1 as follows. Recall that we have constructed a Lagrangian monomorphism $G: T(T^{n-1} \times [0, 1]) \rightarrow \xi$ (see Equation (4.2)). Considering df_i on a neighborhood of $T^{n-1} \times \{i\} \subset T^{n-1} \times [0, 1]$, $i = 0, 1$ respectively, we have a family $F_t: T(T^{n-1} \times [0, 1]) \rightarrow \xi$ of Legendrian monomorphisms that connects $df_0 = F_0$ and $df_1 = F_1$. Setting $X_t := F_t(\partial/\partial s)$, we obtain a required family X_t of vector fields, where s is a coordinate of $[0, 1]$.

From the isotropic isotopy $\tilde{f}_t: T^{n-1} \rightarrow (M, \xi)$ with the isotropic framings X_t , we obtain a family $\tilde{f}_t: T^{n-1} \times [0, 1] \rightarrow (M, \xi)$ of Legendrian embeddings that satisfies $\tilde{f}_t(T^{n-1} \times [0, 1]) = \Lambda_i$ for $i = 0, 1$. In fact, the the given Legendrian submanifolds Λ_i can be shrunk sufficiently to the core

tori $f_i(T^{n-1})$ by a Legendrian isotopy along isotropic vector fields $(f_i)_*(\partial/\partial s)$. Then the family of embeddings of $T^{n-1} \times [0, 1]$ constructed by $\tilde{f}_i(T^{n-1})$ and the framings X_i implies the isotopy of embeddings connecting the shrunk Λ_i . Thus, we obtain a family $\tilde{f}_t: T^{n-1} \times [0, 1] \rightarrow (M, \xi)$ of Legendrian embeddings that satisfies $\tilde{f}_i(T^{n-1} \times [0, 1]) = f_i(T^{n-1} \times [0, 1]) = \Lambda_i$ for $i = 0, 1$.

Step 3: Last of all we extend the isotopy \tilde{f}_t of Legendrian embeddings to an ambient isotopy.

We first review the ambient isotopy theorem (see for example [Ge]). It implies that an isotopy of isotropic embeddings can be extended to an isotopy of global contact diffeomorphisms.

Proposition 4.5. *Let $\psi_t: N \rightarrow (M, \xi)$ be an isotopy of isotropic embeddings of a closed manifold N into a contact manifold (M, ξ) . Then there exists a compactly supported global contact isotopy $\Psi_t: (M, \xi) \rightarrow (M, \xi)$ that restricts to the given isotopy ψ_t . In other words, Ψ_t satisfies $\Psi_0 = \text{id}$, and $\Psi_t \circ \psi_0 = \psi_t$.*

Applying the ambient isotopy theorem, we obtain the isotopy φ_t required in the statement of Lemma 4.4. In fact, we have constructed an isotopy $\tilde{f}_t: T^{n-1} \times [0, 1] \rightarrow (M, \xi)$ of Legendrian embeddings which satisfies $\tilde{f}_0 = f_0$ and $\tilde{f}_1 = f_1$. Then, applying Proposition 4.5, we obtain a compactly supported ambient contact isotopy $\varphi_t: (M, \xi) \rightarrow (M, \xi)$ that satisfies $\varphi_0 = \text{id}$, $\varphi_t \circ \tilde{f}_0 = \tilde{f}_t$. Especially, since $\tilde{f}_i(T^{n-1} \times [0, 1]) = f_i(T^{n-1} \times [0, 1]) = \Lambda_i$ for $i = 0, 1$, it follows that $\varphi_1(\Lambda_0) = \Lambda_1$. Further, since $\tilde{f}_i(T^{n-1}) \subset (M, \xi)$ does not intersect the plastikstufe \mathcal{P} , so does $\tilde{f}_i(T^{n-1} \times [0, 1])$ for any $t \in [0, 1]$. Then φ_t is constructed so that it is identity near \mathcal{P} . Thus, the isotopy φ_t is the required one. Lemma 4.4 has been proved. \square

Now, we show Theorem 4.3.

Proof of Theorem 4.3. The fundamental idea is to apply Theorem 2.3 to a family of isotropic curves simultaneously. In order to do that, we should move a Legendrian submanifold to a suitable position. We need Lemma 4.4 for that.

First, we move the given Legendrian submanifold by Lemma 4.4. Let $\Lambda \subset (M, \xi)$ be a Legendrian submanifold disjoint from the plastikstufe \mathcal{P} . Recall that the plastikstufe $\mathcal{P} \subset (M, \xi)$ has toric core with trivial rotation from the assumption. Then, from Lemma 4.4, we have an isotopy $\varphi_t: (M, \xi) \rightarrow (M, \xi)$ and a Legendrian submanifold $\Lambda_0 \subset \Lambda$ for which $\varphi_1(\Lambda_0)$ lies in the standard tubular neighborhood $U_{\text{PS}} \subset (M, \xi)$ of \mathcal{P} and diffeomorphic to $I \times T_0^{n-1} \subset \mathbb{R}^3 \times T^*T^{n-1}$ by $\varphi_{\text{PS}}: (U_{\text{PS}}, \xi) \rightarrow (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{PS}})$, where $I = (0, \varepsilon) \subset (\mathbb{R}^3, \xi_{\text{ot}})$ is a Legendrian open segment and $T_0^{n-1} \subset T^*T^{n-1}$ is the zero section. We may use the same notation Λ, Λ_0 for the modified Legendrian submanifolds $\varphi_1(\Lambda), \varphi_1(\Lambda_0) \subset (M, \xi)$.

Next, we apply an idea of Theorem 2.3 to the Legendrian submanifold Λ . We discuss in the standard tubular neighborhood $U_{\text{PS}} \subset (M, \xi)$ of the plastikstufe \mathcal{P} with the contact embedding $\varphi_{\text{PS}}: (U_{\text{PS}}, \xi) \rightarrow (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{PS}})$. By taking U_{PS} appropriately small, we may assume $\Lambda \cap U_{\text{PS}} = \Lambda_0$. In other words, we discuss in $(\varphi_{\text{PS}}(U_{\text{PS}}), \xi_{\text{PS}}) \subset (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{PS}})$ and use the same notation Λ_0 for $\varphi_{\text{PS}}(\Lambda_0) \subset (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{PS}})$. Recall that in the standard neighborhood, the plastikstufe \mathcal{P} is $D_{\text{ot}}^2 \times T_0^{n-1} \subset \mathbb{R}^3 \times T^*T^{n-1}$ and that the Legendrian submanifold Λ_0 is disjoint from \mathcal{P} . From the discussion above, Λ_0 is $I \times T_0^{n-1} \subset \mathbb{R}^3 \times T^*T^{n-1}$. Each object \mathcal{P}, Λ_0 is a

direct product, with the zero-section $T_0^{n-1} \subset T^*T^{n-1}$, of an overtwisted disc $D_{\text{ot}}^2 \subset (\mathbb{R}^3, \ker \alpha_{\text{ot}})$ and a Legendrian segment $I \subset (\mathbb{R}^3, \ker \alpha_{\text{ot}})$, respectively. Then, for each $p \in T_0^{n-1}$, we apply Theorem 2.3 simultaneously. We obtain a Legendrian open segment $\tilde{I} \subset (\mathbb{R}^3, \ker \alpha_{\text{ot}})$ which is a negative destabilization of I , and a Legendrian submanifold $\tilde{I} \times T_0^{n-1} \subset (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{PS}})$. Let $\tilde{\Lambda}_0$ denote $\tilde{I} \times T_0^{n-1}$ or the corresponding Legendrian submanifold in (M, ξ) . Comparing $\Lambda_0 \subset \Lambda$ and $\tilde{\Lambda}_0$, we look for a Loose chart for $\Lambda \subset (M, \xi)$.

We look for the loose chart in $(\varphi_{\text{PS}}(U_{\text{PS}}), \xi_{\text{PS}}) \subset (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{PS}})$. Let $U \subset (\mathbb{R}^3, \xi_{\text{ot}} = \ker \alpha_{\text{ot}})$ be an open ball including I, \tilde{I} , and $D^*T^{n-1} \subset T^*T^{n-1}$ a disc-bundle for some metric that satisfy $U \times D^*T^{n-1} \subset \varphi_{\text{PS}}(U_{\text{PS}})$. We find appropriate subsets in both U and D^*T^{n-1} in what follows.

We find a neighborhood of $I \subset U \subset (\mathbb{R}^3, \xi_{\text{ot}})$ contactomorphic to a convex open subset in $(\mathbb{R}^3, \xi_{\text{std}})$ where I is a negative stabilization as follows. From the view point of \tilde{I} , there exists a tubular neighborhood $\tilde{W} \subset (U, \xi_{\text{ot}})$, including I as well after some isotopy, which is contactomorphic to the standard tubular neighborhood W of $\{x = 0, z = 0\} \subset (\mathbb{R}^3, \xi_{\text{std}} = \ker(dz - ydx))$. Note that the image of I in $W \subset (\mathbb{R}^3, \xi_{\text{std}})$ is a negative stabilization of the image of \tilde{I} . We abuse the same notation I, \tilde{I} even in $(\mathbb{R}^3, \xi_{\text{std}})$. Then $W \times D^*T^{n-1} \subset (\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{st}})$ is a tubular neighborhood of $\tilde{I} \times T_0^{n-1}$ which is contactomorphic to a neighborhood of $\tilde{\Lambda}_0 = \tilde{I} \times T_0^{n-1} \subset (U_{\text{PS}}, \xi)$.

Further, we find a sufficiently large, in some sense, subspace in $(D^*T^{n-1}, \lambda_{\text{can}})$. For any metric, the set $V_\rho := \{(\mathbf{p}, \mathbf{q}) \in T^*T^{n-1} \mid |\mathbf{p}| < \rho, |\mathbf{q}| < \rho\}$ is included in D^*T^{n-1} for some $\rho > 0$. Then take a constant $\varepsilon > 0$ so that $\varepsilon < \rho$. For the constant ε , let L_ε be a Legendrian curve in $(\mathbb{R}^3, \xi_{\text{std}})$ defined as

$$\left(\varepsilon t^2, \frac{15}{4}\varepsilon, \frac{\varepsilon^2}{2}(3t^2 - 5t^3) \right),$$

that is isotopic to a negative stabilization of $\{x = 0, z = 0\} \subset (\mathbb{R}^3, \xi_{\text{std}})$. Since L_ε is isotopic to I , $(W \times D^*T^{n-1}, L_\varepsilon \times T_0^{n-1})$ is contactomorphic to $(W \times D^*T^{n-1}, \Lambda_0 = I \times T_0^{n-1})$.

Then the pair $(N, \tilde{\Lambda}_0')$ of submanifolds of (M, ξ) , corresponding to $(W \times V_\rho, L_\varepsilon \times T_0^{n-1})$ in $(\mathbb{R}^3 \times T^*T^{n-1}, \xi_{\text{std}})$ by $\Psi \circ \varphi_{\text{PS}}$, is contactomorphic to a loose chart. In fact, by the contactomorphism

$$h: (\mathbb{R}^3, \xi_{\text{std}}) \rightarrow (\mathbb{R}^3, \xi_{\text{std}}), \quad (x_i, y_i, z) \mapsto (x_i/\varepsilon, y_i/\varepsilon, z/\varepsilon^2),$$

$W \times V_\rho$ and $L_\varepsilon \times T_0^{n-1}$ is mapped to $W' \times V_{\rho/\varepsilon}$ and $L'_\varepsilon \times T_0^{n-1}$ respectively, where $W' \subset \mathbb{R}^3$ is an open set, $L'_\varepsilon \subset (\mathbb{R}^3, \xi_{\text{std}})$ is a Legendrian segment isotopic to L_ε . Since $\varepsilon < \rho$, we have $\rho/\varepsilon > 1$.

Thus, Theorem 4.3 has been proved. \square

4.2.2 Proof of overtwistedness

Now, we show the main issue of this subsection. We show that the existence of a small plastikstufe with toric core and trivial rotation implies overtwistedness.

Proposition 4.6. *Let (M, ξ) be a contact manifold of dimension $2n + 1 > 3$. The contact structure ξ is overtwisted if there exists a small plastikstufe with toric core that has trivial rotation.*

Proof. Assume that there exists a small plastikstufe $\mathcal{P}_{T^{n-1}}$ with toric core and trivial rotation in a contact manifold (M, ξ) of dimension $2n + 1$. Let $\Lambda_0 \subset (M, \xi)$ be the trivial Legendrian sphere (see Subsection 2.3). Applying some isotopy, we may assume that $\Lambda_0 \cap \mathcal{P}_{T^{n-1}} = \emptyset$. Then, from Theorem 4.3, the Legendrian submanifold Λ_0 is loose. By means of Proposition 2.6, the contact structure ξ is overtwisted. \square

5 Modification that creates plastikstufes with toric core

The motivation of this paper is a generalized Lutz twist introduced in [Aj]. The modification makes a contact manifold overtwisted in any dimension. We should note that it is proved directly in [Aj] that the modification makes an S^1 -family of overtwisted discs. On the other hand, using a result in this paper, it is confirmed that the modification makes a contact manifold overtwisted from another point of view. In this section, we find a small plastikstufe with toric core that has trivial rotation in the modification.

First, we roughly recall a generalized Lutz twist introduced in [Aj]. Let (M, ξ) be a contact manifold of dimension $2n + 1$, and $\Gamma \subset (M, \xi)$ an embedded circle transverse to ξ . Along Γ there exists the standard tubular neighborhood contactomorphic to a tubular neighborhood of the transverse circle $S^1 \times \{0\} \subset (S^1 \times \mathbb{R}^{2n}, \xi_0 = \ker \alpha_0)$, where

$$\alpha_0 = d\phi + \sum_{i=1}^n r_i^2 d\theta_i$$

with respect to the cylindrical coordinates (ϕ, r_i, θ_i) of $S^1 \times \mathbb{R}^{2n}$. The generalized Lutz twist is defined as a replacement of such a small tubular neighborhood with the same neighborhood with a certain contact structure, without changing the manifold. The substitute is constructed in [Aj]. Leaving the precise definition to [Aj], we introduce an important part for the construction of the substitute, where we find plastikstufes. Let ζ be the hyperplane field on $S^1 \times \mathbb{R}^{2n}$ defined as $\zeta = \ker \omega_{\text{tw}}$, where

$$\omega_{\text{tw}} = \prod_{i=1}^n (\cos r_i^2) d\phi + \sum_{i=1}^n (\sin r_i^2) d\theta_i,$$

with respect to the cylindrical coordinates (ϕ, r_i, θ_i) of $S^1 \times \mathbb{R}^{2n}$. Unfortunately, it is not a contact structure but a confoliation. In other words, it has a certain non-contact locus. However, it is proved in [Aj] that it can be approximated to a contact structure.

It is proved, in [Aj], that there exists an S^1 -family of plastikstufes with toric core in $(S^1 \times \mathbb{R}^{2n}, \zeta)$. Although it is described, in [Aj], in terms of the so-called bordered Legendrian open book (bLob), a generalization of plastikstufe, it can be translated easily. The object to be observed is the submanifold

$$P := \{(\phi, r_1, \theta_1, \dots, r_n, \theta_n) \in S^1 \times \mathbb{R}^{2n} \mid r_1 \leq \sqrt{\pi}, r_2 = \dots = r_n = \sqrt{\pi}\}.$$

It is diffeomorphic to $S^1 \times D^2 \times T^{n-1}$. From [Aj], P lies in the contact locus of $(S^1 \times \mathbb{R}^{2n}, \zeta)$ and

P is an S^1 -family of bLobs whose bindings are T^{n-1} . We show that each

$$\begin{aligned} P_s &:= \left\{ (\phi, r_i, \theta_i) \in S^1 \times \mathbb{R}^{2n} \mid \phi = s, r_1 \leq \sqrt{\pi}, r_2 = \dots = r_n = \sqrt{\pi} \right\}, \quad s \in S^1, \\ &= \left\{ (s, r_1, \theta_1, \sqrt{\pi}, \theta_2, \dots, \sqrt{\pi}, \theta_n) \in S^1 \times \mathbb{R}^{2n} \mid r_1 \in [0, \sqrt{\pi}], \theta_1, \dots, \theta_n \in S^1 \right\} \end{aligned} \quad (5.1)$$

is a plastikstufe with toric core T^{n-1} in terms of the definition of plastikstufe in Subsection 2.1. In fact, P_s is diffeomorphic to $D^2 \times T^{n-1}$. With respect to this correspondence, a point $z \in D^2$ corresponds to $(r_1^z, \theta_1^z) \in \mathbb{R}^2$. Then the submanifold

$$\{z\} \times T^{n-1} = \left\{ (s, r_1^z, \theta_1^z, \sqrt{\pi}, \theta_2, \dots, \sqrt{\pi}, \theta_n) \in S^1 \times \mathbb{R}^{2n} \mid \theta_2, \dots, \theta_n \in S^1 \right\} \subset P_s$$

is tangent to $\zeta = \ker \omega_{\text{tw}}$, since $\omega_{\text{tw}}|_{T(\{z\} \times T^{n-1})} = (-1)^{n-1} \cos(r_1^z)^2 d\phi$. On the other hand, a point $b \in T^{n-1}$ corresponds to $(\sqrt{\pi}, \theta_2^b, \dots, \sqrt{\pi}, \theta_n^b) \in \mathbb{R}^{2n-2}$. Then on the submanifold

$$D^2 \times \{b\} = \left\{ (s, r_1, \theta_1, \sqrt{\pi}, \theta_2^b, \dots, \sqrt{\pi}, \theta_n^b) \in S^1 \times \mathbb{R}^{2n} \mid r_1 \in (0, 1], \theta_1 \in S^1 \right\} \subset P_s,$$

the 1-form restricts to $\omega_{\text{tw}}|_{T(D^2 \times \{b\})} = (\sin r_1^2) d\theta_1$. Then the singular foliation on $D^2 \times \{b\}$ generated by $\zeta \cap T(D^2 \times \{b\})$ is like Figure 2.1 (2). Thus, we have confirmed that P_s is a plastikstufe.

It is clear that the plastikstufe $P_s \subset S^1 \times \mathbb{R}^{2n}$ is “small.” In fact, P_s lies in the boundary of the polydisc $\{s\} \times (D^2)^n \subset S^1 \times \mathbb{R}^{2n}$. And the entire polydisc is taken in, in the modification defined in [Aj].

It remains to show that the plastikstufe has trivial rotation. According to the Definition 3.3 in Subsection 3.2, we show that the leaf ribbon of the Plastikstufe P_s is Legendrian isotopic to the standard Legendrian $(0, 1) \times T^{n-1}$ in $(S^1 \times \mathbb{R}^{2n}, \zeta)$. A leaf ribbon of P_s is

$$LR_{\theta_1^c} := \left\{ (s, r_1, \theta_1^c, \sqrt{\pi}, \theta_2, \dots, \sqrt{\pi}, \theta_n) \in S^1 \times \mathbb{R}^{2n} \mid r_1 \in (0, \sqrt{\pi}), \theta_2, \dots, \theta_n \in S^1 \right\} \cong (0, 1) \times T^{n-1}$$

for a constant $\theta_1^c \in S^1$ comparing it with P_s in Equation (5.1). Note that $\langle \partial/\partial r_1, \partial/\partial r_2 \rangle$ generates an isotropic plane field for $\zeta = \ker \omega_{\text{tw}}$. Then $LR_{\theta_1^c}$ is Legendrian isotopic to

$$\begin{aligned} \widetilde{LR}_{\theta_1^c} &:= \left\{ (s, \sqrt{\pi}, \theta_1^c, r_2, \theta_2, \sqrt{\pi}, \theta_3, \dots, \sqrt{\pi}, \theta_n) \in S^1 \times \mathbb{R}^{2n} \mid r_2 \in (0, \sqrt{\pi}), \theta_2, \dots, \theta_n \in S^1 \right\} \\ &\cong (0, 1) \times T^{n-1}. \end{aligned}$$

By regarding $\left\{ (s, \sqrt{\pi}, \theta_1^c, r_2, \theta_2, (0, 0), \dots, (0, 0)) \in S^1 \times \mathbb{R}^{2n} \mid r_2 \in [0, \sqrt{\pi}], \theta_2 \in S^1 \right\} = D^2$, the modified leaf ribbon $\widetilde{LR}_{\theta_1^c}$ can be considered as

$$\{s\} \times \left\{ (\sqrt{\pi}, \theta_1^c) \right\} \times (D^2 \setminus \{0\}) \times T^{n-2} \subset (S^1 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{2(n-2)}, \zeta),$$

the standard embedded Legendrian $(0, 1) \times T^{n-1}$. This implies that the plastikstufe P_s has trivial rotation.

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Department of Mathematics,
Hokkaido University,
Sapporo, 060–0810, Japan.

e-mail: j-adachi@math.sci.hokudai.ac.jp